Exercise 6F

1 a $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 4 \\ 1 & 5-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)(5-\lambda) - 4 = 10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6$$

The eigenvalues are the solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Hence:

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 1)(\lambda - 6) = 0$

$\lambda = 1$ or $\lambda = 6$

So the eigenvalues of **A** are 1 and 6.

To find an eigenvector of **A** corresponding to the eigenvalue 1:

$$\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x+4y \\ x+5y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating the upper elements gives:

$2x + 4y = x \Rightarrow x = -4y$

Let $y = 1$, then $x = -4$

Therefore, an eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 6:

$$\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x+4y \\ x+5y \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \end{pmatrix}$$

Equating the upper elements gives:

$2x + 4y = 6x \Rightarrow y = x$

Let $y = 1$, then $x = 1$

Therefore, an eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1 b $\mathbf{A}=\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}4-\lambda & -1\\ -1 & 4-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(4-\lambda)(4-\lambda)-1$$
$$=16-8\lambda+\lambda^2-1$$
$$=\lambda^2-8\lambda+15$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$\lambda^2-8\lambda+15=0$

$(\lambda-3)(\lambda-5)=0$

$\lambda=3$ or $\lambda=5$

So the eigenvalues of **A** are 3 and 5.

To find an eigenvector of **A** corresponding to the eigenvalue 3:

$$\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=3\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}4x-y\\ -x+4y\end{pmatrix}=\begin{pmatrix}3x\\ 3y\end{pmatrix}$$

Equating the upper elements gives:

$4x-y=3x\Rightarrow x=y$

Let $y=1$, then $x=1$

Therefore, an eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ 1\end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 5:

$$\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}4x-y\\ -x+4y\end{pmatrix}=\begin{pmatrix}5x\\ 5y\end{pmatrix}$$

Equating the upper elements gives:

$4x-y=5x\Rightarrow y=-x$

Let $y=1$, then $x=-1$

Therefore, an eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix}-1\\ 1\end{pmatrix}$

1 c $\mathbf{A}=\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}$

$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$=\begin{pmatrix}3-\lambda & -2\\0 & 4-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=(3-\lambda)(4-\lambda)-0$

$=\lambda^2-7\lambda+12$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$\lambda^2-7\lambda+12=0$

$(\lambda-3)(\lambda-4)=0$

$\lambda=3$ or $\lambda=4$

So the eigenvalues of **A** are 3 and 4.

To find an eigenvector of **A** corresponding to the eigenvalue 3:

$\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=3\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}3x-2y\\4y\end{pmatrix}=\begin{pmatrix}3x\\3y\end{pmatrix}$

Equating the upper elements gives:

$3x-2y=3x\Rightarrow y=0$

$\begin{pmatrix}3x\\0\end{pmatrix}=\begin{pmatrix}3x\\0\end{pmatrix}$

Therefore, an eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\0\end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 4:

$\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=4\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}3x-2y\\4y\end{pmatrix}=\begin{pmatrix}4x\\4y\end{pmatrix}$

Equating the upper elements gives:

$3x-2y=4x\Rightarrow x=-2y$

Let $y=1$, then $x=-2$

Therefore, an eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}-2\\1\end{pmatrix}$

2 a $\mathbf{A}=\begin{pmatrix}3 & 4\\ -2 & 9\end{pmatrix}$

$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3 & 4\\ -2 & 9\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$=\begin{pmatrix}3-\lambda & 4\\ -2 & 9-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=(3-\lambda)(9-\lambda)+8$

$=27-12\lambda+\lambda^2+8$

$=\lambda^2-12\lambda+35$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$\lambda^2-12\lambda+35=0$

$(\lambda-5)(\lambda-7)=0$

$\lambda=5$ or $\lambda=7$

So the eigenvalues of **A** are 5 and 7.

2 b $\begin{pmatrix}3 & 4\\ -2 & 9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}3x+4y\\ -2x+9y\end{pmatrix}=\begin{pmatrix}5x\\ 5y\end{pmatrix}$

Equating the upper elements gives:

$3x+4y=5x$

$y=\frac{1}{2}x$

$\begin{pmatrix}3 & 4\\ -2 & 9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=7\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}3x+4y\\ -2x+9y\end{pmatrix}=\begin{pmatrix}7x\\ 7y\end{pmatrix}$

Equating the upper elements gives:

$3x+4y=7x$

$y=x$

3 a $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3-\lambda & 0 & 0 \\ 2 & 4-\lambda & 2 \\ -2 & 0 & 1-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ -2 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 4-\lambda \\ -2 & 0 \end{vmatrix}$$
$$= (3-\lambda)\left[(4-\lambda)(1-\lambda) - 0\right]$$
$$= (3-\lambda)(4-\lambda)(1-\lambda)$$

The eigenvalues are the solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Hence:

$(3-\lambda)(4-\lambda)(1-\lambda) = 0$

Therefore $\lambda = 1, \lambda = 3, \lambda = 4$

To find an eigenvector of **A** corresponding to the eigenvalue 1:

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x+4y+2zy \\ -2x+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements gives:

$3x = x \Rightarrow x = 0$

Equating the middle elements and setting $x = 0$ gives:

$4y + 2z = y \Rightarrow 3y - 2z$

Setting $z = 3$ gives $y = -2$

Therefore, an eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 3:

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x+4y+2z \\ -2x+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the top elements gives:

$3x = 3x \Rightarrow x = 1$

Equating the bottom elements and setting $x = 1$ gives:
$-2 = 2z \Rightarrow z = -1$
Equating the middle elements and setting $x = 1$ and $z = -1$ gives:
$2 + 4y - 2 = 3y \Rightarrow 4y = 3y \Rightarrow y = 0$

Therefore, an eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 4:
$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$\begin{pmatrix} 3x \\ 2x + 4y + 2z \\ -2x + z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$
Equating the top elements gives:
$3x = 4x \Rightarrow x = 0$
Equating the bottom elements and setting $x = 0$ gives:
$z = 4z \Rightarrow z = 0$
Equating the middle elements and setting $x = 0$ and $z = 0$ gives:
$4y = 4y \Rightarrow y = 1$

Therefore, an eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

3 b $\mathbf{A}=\begin{pmatrix}4 & -2 & -4\\ 2 & 3 & 0\\ 2 & -5 & -4\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4 & -2 & -4\\ 2 & 3 & 0\\ 2 & -5 & -4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}4-\lambda & -2 & -4\\ 2 & 3-\lambda & 0\\ 2 & -5 & -4-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(4-\lambda)\begin{vmatrix}3-\lambda & 0\\ -5 & -4-\lambda\end{vmatrix}+2\begin{vmatrix}2 & 0\\ 2 & -4-\lambda\end{vmatrix}-4\begin{vmatrix}2 & 3-\lambda\\ 2 & -5\end{vmatrix}$$
$$=(4-\lambda)\left[(3-\lambda)(-4-\lambda)-0\right]+2\left[2(-4-\lambda)-0\right]-4\left[-10-2(3-\lambda)\right]$$
$$=-(4-\lambda)(3-\lambda)(4+\lambda)-4(4+\lambda)+40+8(3-\lambda)$$
$$=-(16-\lambda^2)(3-\lambda)-16-4\lambda+40+24-8\lambda$$
$$=-(48-16\lambda-3\lambda^2+\lambda^3)-12\lambda+48$$
$$=-48+16\lambda+3\lambda^2-\lambda^3-12\lambda+48$$
$$=4\lambda+3\lambda^2-\lambda^3$$
$$=\lambda(4+3\lambda-\lambda^2)$$
$$=\lambda(4-\lambda)(1+\lambda)$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:
$\lambda(4-\lambda)(1+\lambda)=0$

Therefore $\lambda=0$, $\lambda=-1$, $\lambda=4$

To find an eigenvector of **A** corresponding to the eigenvalue –1:

$$\begin{pmatrix}4 & -2 & -4\\ 2 & 3 & 0\\ 2 & -5 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-1\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$
$$\begin{pmatrix}4x-2y-4z\\ 2x+3y\\ 2x-5y-4z\end{pmatrix}=\begin{pmatrix}-x\\ -y\\ -z\end{pmatrix}$$

Equating the middle elements gives:
$2x=-4y\Rightarrow x=-2y$

Setting $y=1$ gives $x=-2$

Equating the top elements and setting $x=-2$ and $y=1$ gives:
$-8-2-4z=2\Rightarrow 4z=-12\Rightarrow z=-3$

Therefore, an eigenvector corresponding to the eigenvalue –1 is $\begin{pmatrix}-2\\ 1\\ -3\end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 0:

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the middle elements gives:

$2x = -3y$

Setting $y = 2$ gives $x = -3$

Equating the top elements and setting $x = -3$ and $y = 2$ gives:

$-12 - 4 - 4z = 0 \Rightarrow 4z = -16 \Rightarrow z = -4$

Therefore, an eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 4:

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the middle elements gives:

$2x = y$

Setting $x = 1$ gives $y = 2$

Equating the top elements and setting $x = 1$ and $y = 2$ gives:

$-4z = 4 \Rightarrow z = -1$

Therefore, an eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

4 a $\mathbf{A}=\begin{pmatrix}2&2&-2\\-3&2&0\\1&4&-3\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&2&-2\\-3&2&0\\1&4&-3\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}2-\lambda&2&-2\\-3&2-\lambda&0\\1&4&-3-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)\begin{vmatrix}2-\lambda&0\\4&-3-\lambda\end{vmatrix}-2\begin{vmatrix}-3&0\\1&-3-\lambda\end{vmatrix}-2\begin{vmatrix}-3&2-\lambda\\1&4\end{vmatrix}$$

$$=(2-\lambda)\left[(2-\lambda)(-3-\lambda)-0\right]-2\left[-3(-3-\lambda)-0\right]-2\left[-12-1(2-\lambda)\right]$$

$$=-(2-\lambda)(2-\lambda)(3+\lambda)-6(3+\lambda)+24+2(2-\lambda)$$

$$=-(4-4\lambda+\lambda^2)(3+\lambda)-18-6\lambda+24+4-2\lambda$$

$$=-(12-12\lambda+3\lambda^2+4\lambda-4\lambda^2+\lambda^3)-8\lambda+10$$

$$=-12+12\lambda-3\lambda^2-4\lambda+4\lambda^2-\lambda^3-8\lambda+10$$

$$=-2+\lambda^2-\lambda^3$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$-2+\lambda^2-\lambda^3=0$

Try $\lambda=-1$

$-2+1+1=0$

Therefore, $\lambda+1$ is a factor of $-2+\lambda^2-\lambda^3=0$

$$\begin{array}{r} -\lambda^2+2\lambda-2 \\ \lambda+1\overline{)-\lambda^3+\lambda^2+0\lambda-2} \\ \underline{-\lambda^3-\lambda^2} \\ 2\lambda^2+0\lambda \\ \underline{2\lambda^2+2\lambda} \\ -2\lambda-2 \\ \underline{-2\lambda-2} \\ 0 \end{array}$$

Hence:

$(\lambda+1)(-\lambda^2+2\lambda-2)=0$

$-\lambda^2+2\lambda-2=0$

$\lambda^2-2\lambda+2=0$

$(\lambda-1)^2-1+2=0$

$(\lambda-1)^2=-1$

$\lambda-1=\pm\sqrt{-1}$

Therefore, $\lambda=-1$ is the only real eigenvalue of **A**.

4 b To find an eigenvector of **A** corresponding to the eigenvalue –1:

$$\begin{pmatrix} 2 & 2 & -2 \\ -3 & 2 & 0 \\ 1 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y-2z \\ -3x+2y \\ x+4y-3z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

Equating the middle elements gives:

$3y = 3x \Rightarrow y = x$

Setting $y = 1$ gives $y = 1$

Equating the top elements and setting $x = 1$ and $y = 1$ gives:

$2+2-2z = -1 \Rightarrow z = \frac{5}{2}$

Therefore, an eigenvector corresponding to the eigenvalue –1 is $\begin{pmatrix} 1 \\ 1 \\ \frac{5}{2} \end{pmatrix}$

5 a $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 0 \end{pmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & -1 & 3 \\ 0 & 2-\lambda & 4 \\ 0 & 2 & -\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 2 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ 0 & -\lambda \end{vmatrix} + 3\begin{vmatrix} 0 & 2-\lambda \\ 0 & 2 \end{vmatrix}$$

$$= (2-\lambda)[-\lambda(2-\lambda)-8] + 1(0-0) + 3(0-0)$$

$$= (2-\lambda)(\lambda^2 - 2\lambda - 8)$$

$$= (2-\lambda)(\lambda-4)(\lambda+2)$$

The eigenvalues are the solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Hence:

$(2-\lambda)(\lambda-4)(\lambda+2) = 0$

Therefore $\lambda = -2$, $\lambda = 2$, $\lambda = 4$

5 b To find an eigenvector of **A** corresponding to the eigenvalue 4:

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x - y + 3z \\ 2y + 4z \\ 2y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the bottom elements gives:

$2y = 4z \Rightarrow y = 2z$

Setting $z = 1$ gives $y = 2$

Equating the top elements and setting $z = 1$ and $y = 2$ gives:

$2x - 2 + 3 = 4x \Rightarrow x = \frac{1}{2}$

Therefore, an eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$

6 a $\mathbf{A}=\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(1-\lambda)\begin{vmatrix}4-\lambda&-1\\4&3-\lambda\end{vmatrix}-1\begin{vmatrix}2&-1\\4&3-\lambda\end{vmatrix}+3\begin{vmatrix}2&4-\lambda\\4&4\end{vmatrix}$$
$$=(1-\lambda)\left[(4-\lambda)(3-\lambda)+4\right]-1\left[2(3-\lambda)+4\right]+3\left[8-4(4-\lambda)\right]$$
$$=(1-\lambda)\left[12-7\lambda+\lambda^2+4\right]-1(6-2\lambda+4)+3(8-16+4\lambda)$$
$$=(1-\lambda)(\lambda^2-7\lambda+16)+2\lambda-10+3(4\lambda-8)$$
$$=\lambda^2-7\lambda+16-\lambda^3+7\lambda^2-16\lambda+2\lambda-10+12\lambda-24$$
$$=-\lambda^3+8\lambda^2-9\lambda-18$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$-\lambda^3+8\lambda^2-9\lambda-18=0$

Since 3 is an eigenvalue of **A**, $\lambda-3$ is a factor of $-\lambda^3+8\lambda^2-9\lambda-18$

$$\begin{array}{r|l} & \lambda^2-5\lambda-6 \\ \lambda-3 & \lambda^3-8\lambda^2+9\lambda+18 \\ & \underline{\lambda^3-3\lambda^2} \\ & \quad -5\lambda^2+9\lambda \\ & \quad \underline{-5\lambda^2+15\lambda} \\ & \qquad -6\lambda+18 \\ & \qquad \underline{-6\lambda+18} \\ & \qquad\qquad 0 \end{array}$$

Hence:

$$\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2-5\lambda-6)$$
$$=(\lambda-3)(\lambda-6)(\lambda+1)$$

Therefore $\lambda=-1,\ \lambda=3,\ \lambda=6$

6 b To find an eigenvector of **A** corresponding to the eigenvalue –1:

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

Equating the top elements gives:

$2x+y+3z=0$ **(1)**

Equating the middle elements gives:

$2x+5y-z=0$ **(2)**

Subtracting **(1)** from **(2)** gives:

$4y-4z=0 \Rightarrow y=z$

Setting $y = 1$ gives $z = 1$

Substituting $y = 1$ and $z = 1$ into **(1)** gives:

$2x+1+3=0 \Rightarrow x=-2$

Therefore, an eigenvector corresponding to the eigenvalue –1 is $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 3:

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the top elements gives:

$-2x+y+3z=0$ **(1)**

Equating the middle elements gives:

$2x+y-z=0$ **(2)**

Adding **(1) to (2)** gives:

$2y+2z=0 \Rightarrow y=-z$

Setting $y = 1$ gives $z = –1$

Substituting $y = 1$ and $z = –1$ into **(1)** gives:

$-2x+1-3=0 \Rightarrow x=-1$

Therefore, an eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 6:

$$\begin{pmatrix}1 & 1 & 3\\ 2 & 4 & -1\\ 4 & 4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=6\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}x+y+3z\\ 2x+4y-z\\ 4x+4y+3z\end{pmatrix}=\begin{pmatrix}6x\\ 6y\\ 6z\end{pmatrix}$$

Equating the top elements gives:
$-5x+y+3z=0$ **(1)**
Equating the middle elements gives:
$2x-2y-z=0$ **(2)**
Adding 2 × **(1)** and 5 × **(2)** gives:
$-8y+z=0\Rightarrow z=8y$
Setting $y=1$ gives $z=8$
Substituting $y=1$ and $z=8$ into **(1)** gives:
$-5x+1+24=0\Rightarrow x=5$

Therefore, an eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix}5\\ 1\\ 8\end{pmatrix}$

7 a $\mathbf{A}=\begin{pmatrix}2 & 2 & 1\\ -2 & 4 & 0\\ 4 & 2 & 5\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2 & 2 & 1\\ -2 & 4 & 0\\ 4 & 2 & 5\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2-\lambda & 2 & 1\\ -2 & 4-\lambda & 0\\ 4 & 2 & 5-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)\begin{vmatrix}4-\lambda & 0\\ 2 & 5-\lambda\end{vmatrix}-2\begin{vmatrix}-2 & 0\\ 4 & 5-\lambda\end{vmatrix}+1\begin{vmatrix}-2 & 4-\lambda\\ 4 & 2\end{vmatrix}$$

$$=(2-\lambda)\left[(4-\lambda)(5-\lambda)-0\right]-2\left[-2(5-\lambda)-0\right]+1\left[-4-4(4-\lambda)\right]$$

$$=(2-\lambda)\left(20-9\lambda+\lambda^2\right)-2(2\lambda-10)+4\lambda-20$$

$$=40-18\lambda+2\lambda^2-20\lambda+9\lambda^2-\lambda^3-4\lambda+20+4\lambda-20$$

$$=40-38\lambda+11\lambda^2-\lambda^3$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:
$\lambda^3-11\lambda^2+38\lambda-40=0$
If 2 is an eigenvalue of **A**, then f(2) = 0
$(2)^3-11(2)^2+38(2)-40=0$
$8-44+76-40=0$
$0=0$
Hence 2 is an eigenvalue of **A**.

7 b Since $(x-2)$ is a factor of **A**:

$$
\begin{array}{r}
\lambda^2 - 9\lambda + 20 \\
\lambda - 2 \overline{\big)\, \lambda^3 - 11\lambda^2 + 38\lambda - 40} \\
\underline{\lambda^3 - 2\lambda^2} \qquad\qquad \\
-9\lambda^2 + 38\lambda \qquad \\
\underline{-9\lambda^2 + 18\lambda} \qquad \\
20\lambda - 40 \\
\underline{20\lambda - 40} \\
0
\end{array}
$$

Hence:

$$\lambda^3 - 11\lambda^2 + 38\lambda - 40 = (\lambda-2)(\lambda^2 - 9\lambda + 20)$$
$$= (\lambda-2)(\lambda-4)(\lambda-5)$$

Therefore $\lambda = 2$, $\lambda = 4$, $\lambda = 5$

c To find an eigenvector of **A** corresponding to the eigenvalue 2:

$$\begin{pmatrix} 2 & 2 & 1 \\ -2 & 4 & 0 \\ 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y+z \\ -2x+4y \\ 4x+2y+5z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the middle elements gives:

$-2x + 4y = 2y \Rightarrow x = y$

Setting $x = 1$ gives $y = 1$

Equating the elements of the top row and substituting $x = 1$ and $y = 1$ gives:

$2 + z = 0 \Rightarrow z = -2$

Therefore, an eigenvector of **A** corresponding to the eigenvalue 2 is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

The magnitude of this eigenvector is:

$\sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

and a normalised eigenvector of **A** corresponding to the eigenvalue 2 is:

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

8 a $\mathbf{A}=\begin{pmatrix}4 & 2 & 1\\ -2 & 0 & 5\\ 0 & 3 & 4\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4 & 2 & 1\\ -2 & 0 & 5\\ 0 & 3 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}4-\lambda & 2 & 1\\ -2 & -\lambda & 5\\ 0 & 3 & 4-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(4-\lambda)\begin{vmatrix}-\lambda & 5\\ 3 & 4-\lambda\end{vmatrix}-2\begin{vmatrix}-2 & 5\\ 0 & 4-\lambda\end{vmatrix}+1\begin{vmatrix}-2 & -\lambda\\ 0 & 3\end{vmatrix}$$
$$=(4-\lambda)\left[-\lambda(4-\lambda)-15\right]-2\left[-2(4-\lambda)-0\right]+1(-6+0)$$
$$=(4-\lambda)(\lambda^2-4\lambda-15)-2(2\lambda-8)-6$$
$$=4\lambda^2-16\lambda-60-\lambda^3+4\lambda^2+15\lambda-4\lambda+16-6$$
$$=-\lambda^3+8\lambda^2-5\lambda-50$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$\lambda^3-8\lambda^2+5\lambda+50=0$

If –2 is an eigenvalue of **A**, then f(–2) = 0

$(-2)^3-8(-2)^2+5(-2)+50=0$

$-8-32-10+50=0$

$0=0$

Hence –2 is an eigenvalue of **A**.

Since –2 is an eigenvalue of **A**, $\lambda+2$ is a factor of $\lambda^3-8\lambda^2+5\lambda+50$

$$\begin{array}{r} \lambda^2-10\lambda+25 \\ \lambda+2\overline{)\lambda^3-8\lambda^2+5\lambda+50} \\ \underline{\lambda^3+2\lambda^2} \\ -10\lambda^2+5\lambda \\ \underline{-10\lambda^2-20\lambda} \\ 25\lambda+50 \\ \underline{25\lambda+50} \\ 0 \end{array}$$

Hence:

$$\lambda^3-8\lambda^2+5\lambda+50=(\lambda+2)(\lambda^2-10\lambda+25)$$
$$=(\lambda+2)(\lambda-5)^2$$

Therefore $\lambda=-2$, $\lambda=5$

Hence there is only one eigenvalue other than –2

8 b To find an eigenvector of **A** corresponding to the eigenvalue –2:

$$\begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x+2y+z \\ -2x+5z \\ 3y+4z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

Equating the bottom elements gives:

$3y+4z=-2z \Rightarrow y=-2z$

Setting $z = 1$ gives $y = -2$

Equating the top elements and setting $y = -2$ and $z = 1$ gives:

$4x-4+1=-2x \Rightarrow x=\frac{1}{2}$

Therefore, an eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} \frac{1}{2} \\ -2 \\ 1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 5:

$$\begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x+2y+z \\ -2x+5z \\ 3y+4z \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \\ 5z \end{pmatrix}$$

Equating the bottom elements gives:

$3y+4z=5z \Rightarrow 3y=z$

Setting $y = 1$ gives $z = 3$

Equating the top elements and setting $y = 1$ and $z = 3$ gives:

$4x+2+3=5x \Rightarrow x=5$

Therefore, an eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$

9 a $\mathbf{A}=\begin{pmatrix}1 & -1 & 0\\ -1 & 0 & 1\\ 1 & 2 & 1\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1 & -1 & 0\\ -1 & 0 & 1\\ 1 & 2 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & -1 & 0\\ -1 & -\lambda & 1\\ 1 & 2 & 1-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(1-\lambda)\begin{vmatrix}-\lambda & 1\\ 2 & 1-\lambda\end{vmatrix}+1\begin{vmatrix}-1 & 1\\ 1 & 1-\lambda\end{vmatrix}+0\begin{vmatrix}-1 & -\lambda\\ 1 & 2\end{vmatrix}$$

$$=(1-\lambda)\left[-\lambda(1-\lambda)-2\right]+1\left[-1(1-\lambda)-1\right]+0(-2+\lambda)$$

$$=(1-\lambda)(\lambda^2-\lambda-2)+\lambda-2$$

$$=(1-\lambda)(\lambda-2)(\lambda+1)+\lambda-2$$

$$=(\lambda-2)\left[(1-\lambda)(\lambda+1)+1\right]$$

$$=(\lambda-2)(1-\lambda^2+1)$$

$$=(\lambda-2)(2-\lambda^2)$$

$$=(\lambda-2)(\sqrt{2}-\lambda)(\sqrt{2}+\lambda)$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$(\lambda-2)(\sqrt{2}-\lambda)(\sqrt{2}+\lambda)=0$

Therefore $\lambda=2,\ \lambda=\pm\sqrt{2}$

9 b To find an eigenvector of **A** corresponding to the eigenvalue 2:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements gives:

$x - y = 2x \Rightarrow x = -y$

Setting $x = 1$ gives $y = -1$

Equating the bottom elements and setting $x = 1$ and $y = -1$ gives:

$1 - 2 + z = 2z \Rightarrow z = -1$

Therefore, an eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue $-\sqrt{2}$:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\sqrt{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} -\sqrt{2}x \\ -\sqrt{2}y \\ -\sqrt{2}z \end{pmatrix}$$

Equating the top elements gives:

$x - y = -\sqrt{2}x \Rightarrow x\left(1+\sqrt{2}\right) = y$

Setting $x = 1$ gives $y = 1+\sqrt{2}$

Equating the middle elements and setting $x = 1$ and $y = 1+\sqrt{2}$ gives:

$-1 + z = -\sqrt{2}\left(1+\sqrt{2}\right) \Rightarrow z = 1-\left(\sqrt{2}+2\right) \Rightarrow z = -1-\sqrt{2}$

Therefore, an eigenvector corresponding to the eigenvalue $-\sqrt{2}$ is $\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ -1-\sqrt{2} \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue $\sqrt{2}$:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} \sqrt{2}x \\ \sqrt{2}y \\ \sqrt{2}z \end{pmatrix}$$

Equating the top elements gives:

$x - y = \sqrt{2}x \Rightarrow x\left(1-\sqrt{2}\right) = y$

Setting $x = 1$ gives $y = 1-\sqrt{2}$

Equating the middle elements and setting $x = 1$ and $y = 1-\sqrt{2}$ gives:

$-1 + z = \sqrt{2}\left(1-\sqrt{2}\right) \Rightarrow z = 1 + \left(\sqrt{2}-2\right) \Rightarrow z = \sqrt{2}-1$

Therefore, an eigenvector corresponding to the eigenvalue $\sqrt{2}$ is $\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ \sqrt{2}-1 \end{pmatrix}$

10 a $\mathbf{A} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}$

To find an eigenvalue, p, of **A** corresponding to the eigenvector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$:

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = p\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 8+2-2 \\ 2+2a \\ -2+2-b \end{pmatrix} = \begin{pmatrix} 2p \\ 2p \\ -p \end{pmatrix}$$

Equating the top elements gives:

$8+2-2 = 2p \Rightarrow p = 4$

10 b $\begin{pmatrix} 8+2-2 \\ 2+2a \\ -2+2-b \end{pmatrix} = \begin{pmatrix} 2p \\ 2p \\ -p \end{pmatrix}$

$$\begin{pmatrix} 8 \\ 2+2a \\ -b \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ -4 \end{pmatrix}$$

Equating the middle elements gives:

$2+2a = 8 \Rightarrow a = 3$

Equating the bottom elements gives:

$-b = -4 \Rightarrow b = 4$

10 c $\mathbf{A}=\begin{pmatrix}4&1&2\\1&3&0\\-1&1&4\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4&1&2\\1&3&0\\-1&1&4\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}4-\lambda&1&2\\1&3-\lambda&0\\-1&1&4-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(4-\lambda)\begin{vmatrix}3-\lambda&0\\1&4-\lambda\end{vmatrix}-1\begin{vmatrix}1&0\\-1&4-\lambda\end{vmatrix}+2\begin{vmatrix}1&3-\lambda\\-1&1\end{vmatrix}$$
$$=(4-\lambda)[(3-\lambda)(4-\lambda)-0]-1[1(4-\lambda)-0]+2[1+1(3-\lambda)]$$
$$=(3-\lambda)(16-8\lambda+\lambda^2)+\lambda-4+2(4-\lambda)$$
$$=48-24\lambda+3\lambda^2-16\lambda+8\lambda^2-\lambda^3-\lambda+4$$
$$=52-41\lambda+11\lambda^2-\lambda^3$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$52-41\lambda+11\lambda^2-\lambda^3=0$

Since 4 is an eigenvalue of **A**, $\lambda-4$ is a factor of $\lambda^3-11\lambda^2+41\lambda-52$

$$\begin{array}{r} \lambda^2-7\lambda+13 \\ \lambda-4\,\overline{)\,\lambda^3-11\lambda^2+41\lambda-52} \\ \underline{\lambda^3-4\lambda^2} \\ -7\lambda^2+41\lambda \\ \underline{-7\lambda^2+28\lambda} \\ 13\lambda-52 \\ \underline{13\lambda-52} \\ 0 \end{array}$$

Hence:

$\lambda^3-11\lambda^2+41\lambda-52=(\lambda-4)(\lambda^2-7\lambda+13)$

$\lambda^2-7\lambda+13=0$

$\left(\lambda-\frac{7}{2}\right)^2-\frac{49}{4}+13=0$

$\left(\lambda-\frac{7}{2}\right)^2+\frac{3}{4}=0$

$\left(\lambda-\frac{7}{2}\right)^2=-\frac{3}{4}$

$\lambda-\frac{7}{2}=\pm\frac{\sqrt{-3}}{2}$

Hence **A** has only one real eigenvalue.

Challenge

$$\mathbf{A}=\begin{pmatrix}-1 & 0\\ -2 & 1\end{pmatrix}$$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}-1 & 0\\ -2 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}-1-\lambda & 0\\ -2 & 1-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=-(1+\lambda)(1-\lambda)-0$$
$$=-(1-\lambda^2)$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$\lambda^2-1=0$

$\lambda=\pm1$

So the eigenvalues of **A** are –1 and 1.

To find an eigenvector of **A** corresponding to the eigenvalue 1:

$$\begin{pmatrix}-1 & 0\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=1\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}-x\\ -2x+y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

Equating the top elements gives:

$-x=x\Rightarrow x=0$

when $x=0$, $y=y\Rightarrow y=1$

Therefore, an eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix}0\\ 1\end{pmatrix}$ and hence all the points on the y-axis are invariant.

To find an eigenvector of **A** corresponding to the eigenvalue –1:

$$\begin{pmatrix}-1 & 0\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-1\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}-x\\ -2x+y\end{pmatrix}=\begin{pmatrix}-x\\ -y\end{pmatrix}$$

Equating the lower elements gives:

$-2x+y=-y\Rightarrow x=y$

Setting $x = 1$, gives $y = 1$

Therefore, an eigenvector corresponding to the eigenvalue –1 is $\begin{pmatrix}1\\ 1\end{pmatrix}$ and hence all lines parallel to $y=x$ stay parallel to $y = x$ under T. Since every line will cross the y-axis at one point, and this point is invariant under T, every line of the form $y = x + k$ is an invariant line of T, and there are infinitely many of these.